Exchange economy social welfare exercise

Consider an exchange economy with two goods and two agents whose preferences are determined by the following utility functions:

$$u^1(x,y) = \sqrt{xy} + \ln x + \ln y$$
$$u^2(x,y) = -4/x - 1/y$$

and the total resources are

$$\omega^1 + \omega^2 = (3,6)$$

1. Prove that the allocation $x^1 = (1,4)$, $x^2 = (2,2)$ is Pareto efficient.
2. Suppose we use the social welfare function $W = au^1 + bu^2$. Determine weights a, b such that the social welfare function W would choose the previous allocation.

Solution

1. There are two conditions for Pareto efficiency:
   (a) Optimal Consumption Condition:
   $$\frac{\frac{\partial u^1}{\partial x_1}}{\frac{\partial u^1}{\partial y_1}} = \frac{\frac{\partial u^2}{\partial x_2}}{\frac{\partial u^2}{\partial y_2}}$$
   (b) Feasibility Condition:
   $$x_1 + x_2 = 3$$
   $$y_1 + y_2 = 6$$
   This last one is trivially true:
   $$1 + 2 = 3$$
   $$4 + 2 = 6$$
   Let's check the first one:

   We write the marginal rate of substitution for agent 1:
   $$\frac{\partial u^1}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} + \frac{1}{x}, \quad \frac{\partial u^1}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} + \frac{1}{y}$$
   from which
   $$RMS^1(1,4) = \frac{2}{1/2} = 4$$
   and the marginal rate of substitution for agent 2:
   $$\frac{\partial u^2}{\partial x} = \frac{4}{x^2}, \quad \frac{\partial u^2}{\partial y} = \frac{1}{y^2}$$
   from which
   $$tealRMS^2(2,2) = 4$$
   so the allocation $x^1 = (1,4)$, $x^2 = (2,2)$ is Pareto efficient.

2. The central planner's problem can be formulated as:
   $$\max_{x_1,y_1,x_2,y_2} W = a\left(\sqrt{x_1 y_1} + \ln x_1 + \ln y_1\right) + b\left(-\frac{4}{x_2} - \frac{1}{y_2}\right)$$
   subject to the resource constraints:
   $$x_1 + x_2 \leq 3$$
   $$y_1 + y_2 \leq 6$$
   We can introduce Lagrange multipliers λ and μ for the resource constraints and write the Lagrangian:
   $$\mathcal{L} = a\left(\sqrt{x_1 y_1} + \ln x_1 + \ln y_1\right) + b\left(-\frac{4}{x_2} - \frac{1}{y_2}\right) + \lambda(3 - x_1 - x_2) + \mu(6 - y_1 - y_2)$$

The first-order conditions for a maximum are obtained by differentiating the Lagrangian with respect to $x_1, y_1, x_2, y_2, \lambda$, and μ:

$$\frac{\partial \mathcal{L}}{\partial x_1} = a\left(\frac{\sqrt{y_1}}{2\sqrt{x_1}} + \frac{1}{x_1}\right) - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y_1} = a\left(\frac{\sqrt{x_1}}{2\sqrt{y_1}} + \frac{1}{y_1}\right) - \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = b\frac{4}{x_2^2} - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y_2} = b\frac{1}{y_2} - \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 3 - x_1 - x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \mu} = 6 - y_1 - y_2 = 0$$

Solving the first 4 equations:

$$a\left(\frac{\sqrt{y_1}}{2\sqrt{x_1}} + \frac{1}{x_1}\right) = \lambda$$
$$a\left(\frac{\sqrt{x_1}}{2\sqrt{y_1}} + \frac{1}{y_1}\right) = \mu$$
$$b\frac{4}{x_2^2} = \lambda$$

$$b\frac{1}{y_2^2} = \mu$$

We obtain:

$$a\left(\frac{\sqrt{y_1}}{2\sqrt{x_1}} + \frac{1}{x_1}\right) = b\frac{4}{x_2^2}$$
$$a\left(\frac{\sqrt{x_1}}{2\sqrt{y_1}} + \frac{1}{y_1}\right) = b\frac{1}{y_2^2}$$

Using the allocation:

$$a\left(\frac{\sqrt{4}}{2\sqrt{1}} + \frac{1}{1}\right) = b\frac{4}{4}$$

$$a\left(\frac{\sqrt{1}}{2\sqrt{4}} + \frac{1}{4}\right) = b\frac{1}{4}$$

$$2a = b$$

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We can choose for example $a = 2$ and $b = 4$